By the proposed method or hints (if any), or otherwise, evaluate each of the following integrals, giving all your answers in exact forms wherever appropriate.

1. $\int \frac{\ln x}{x}\, dx$ [3]

$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$

$= \int u \cdot du = \frac{1}{2}u^2 + c$

$= \frac{1}{2}(\ln x)^2 + c$

2. $\int \frac{1}{x^3 - x}\, dx$; by first decomposing into partial fractions [5]

$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

$1 = A(x+1)(x-1) + Bx(x-1) + C\cdot x\cdot(x+1)$

$x=0, \Rightarrow A=-1, \quad x=1, \Rightarrow C=\frac{1}{2} \quad x=-1, \Rightarrow B=\frac{1}{2}$

$\Rightarrow \int \frac{1}{x^3-x}\, dx = \int \frac{-1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}\, dx$

$= -\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$

3. $\int_2^3 \frac{x}{x^2+1}\,dx$; by substitution [5]

$= \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_2^3$

$= \frac{1}{2}\left(\left[\ln 10\right]-\left[\ln 5\right]\right)$

$= \frac{1}{2}\ln 2$

4. $\int_2^3 \frac{x}{x^2-1}\,dx$ [5]

$= \frac{1}{2}\int_2^3 \frac{2x}{x^2-1}\,dx$

$= \frac{1}{2}\left[\ln|x^2-1|\right]_2^3$

$= \frac{1}{2}\left(\left[\ln 8\right]-\left[\ln 3\right]\right)$

$= \frac{1}{2}\ln\frac{8}{3}$

5. $\int_0^4 x\sqrt{2x+1}\,\mathrm{d}x$ [5]

$u = \sqrt{2x+1}$ $\quad u^2 = 2x+1$ $\quad x = \frac{u^2-1}{2}$

$x=0,\ u=1$ $\quad \frac{dx}{du} = u$

$x=4,\ u=3$

$= \int_1^3 \frac{u^2-1}{2} \cdot u\, u\, du$

$= \frac{1}{2}\int_1^3 (u^4-u^2)\,du = \frac{1}{2}\left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^3$

$= \frac{1}{2}\left(\frac{198}{5} + \frac{2}{15}\right) = \frac{298}{15}$

6. $\int \frac{x}{x^2+x+1}\,\mathrm{d}x;$ [5]

$\int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{x^2+x+1}\,dx$

$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx$

$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2 + 1}\,dx$

$= \frac{1}{2}\ln|x^2+x+1| - \frac{2}{3}\cdot\sqrt{\frac{3}{4}}\ \tan^{-1}\left(\frac{2\sqrt{3}}{3}(x+\frac{1}{2})\right) + C$

$= \frac{1}{2}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}(x+\frac{1}{2})\right) + C$

7. $\int_0^{\pi^2} \sin\sqrt{x}\,\mathrm{d}x$; by using a suitable substitution, followed by integration by parts [6]

$u = \sqrt{x}$. $x = 0,\ u = 0$ $x = \pi^2$ $u = \pi$

$u^2 = x$ $\frac{dx}{du} = 2u$ dx

$\int_0^{\pi} \sin u \cdot 2u\, du = 2u(-\cos u) - \int_0^{\pi} (-2\cos u)\, du$

$2u$ $\sin u$

2 $-\cos u$ $= \left[-2u\cos u + 2\sin u\right]_0^{\pi}$

$= [\,2\pi\,] - [0] = 2\pi$

8. $\int_1^2 \frac{\mathrm{e}^{\frac{1}{x}}}{x^3}\,\mathrm{d}x$ [6]

$u = \frac{1}{x}$ $\frac{du}{dx} = -x^{-2}$ $du = -x^{-2}dx$

$x = 1,\ u = 1$

$x = 2,\ u = \frac{1}{2}$

$= \int_1^{\frac{1}{2}} e^u \cdot u^3 \cdot (-u^{-2})\, du$

$= \int_{\frac{1}{2}}^{1} u e^u\, du = \left[u e^u - e^u\right]_{\frac{1}{2}}^{1}$

u e^u $= [e - e] - [\frac{1}{2}e^{\frac{1}{2}} - e^{\frac{1}{2}}]$

1 e^u

$= \frac{1}{2}e^{\frac{1}{2}}$

9. $\int e^{-x}\sin 2x\,dx$; by applying integration by parts twice [6]

$u = e^{-x}$ $v' \sin 2x$

$u' = -e^{-x}$ $v = -\frac{1}{2}\cos 2x$

$I = e^{-x}(-\frac{1}{2})\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\,dx$

$u = e^{-x}$ $v' = \cos 2x$

$u' = -e^{-x}$ $v = \frac{1}{2}\sin 2x$

$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\left(\frac{1}{2}e^{-x}\sin 2x + \int e^{-x}\frac{1}{2}\sin 2x\right)$

$I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}I + C$

$\Rightarrow I = \frac{4}{5}\left(-\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x\right) + C$

10. $\int_0^{\frac{\pi}{4}}\sin x\cos 2x\,dx$ $= -\frac{1}{5}e^{-x}\sin 2x - \frac{2}{5}e^{-x}\cos 2x + C$ [6]

$u = \cos x$ $2\cos^2 x - 1$ $\frac{du}{dx} = -\sin x$

$x = 0,\ u = 1$

$x = \frac{\pi}{4},\ u = \frac{\sqrt{2}}{2}$

$-\int_1^{\frac{\sqrt{2}}{2}}(2u^2-1)\,du$

$= \int_{\frac{\sqrt{2}}{2}}^{1}(2u^2-1)\,du$

$= \left[2\times\frac{u^3}{3} - u\right]_{\frac{\sqrt{2}}{2}}^{1} = \left[\frac{2}{3} - 1\right] - \left[\frac{2}{3}\times\frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right]$

$= -\frac{1}{3} + \frac{\sqrt{2}}{3}$

11. $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^3+x^2-x+1}{x^4-1}\,\mathrm{d}x$; by first decomposing into partial fractions [8]

$$\frac{x^3+x^2-x+1}{(x^2-1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$x^3+x^2-x+1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$x=1$

$$2 = A\cdot 2\cdot 2 + 0 \Rightarrow A = \tfrac{1}{2}$$

$x=-1$

$-1+1+1+1$

$$2 = 0 + B(-2)(2) + 0 \Rightarrow B = -\tfrac{1}{2}$$

$x=0$

$$1 = \tfrac{1}{2}(1)(1) - \tfrac{1}{2}(-1)(1) + D(-1) \Rightarrow D=0$$

$x=2$

$$8+4-2+1 = \cancel{\tfrac{1}{2}(3)(5) - \tfrac{1}{2}(1)}(5) + (2C)(3)$$

$$11 = 5 + 6C \Rightarrow C = 1$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} + \frac{x}{x^2+1}\,dx$$

$$= \left[\tfrac{1}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x^2+1|\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}\left(\ln\tfrac{1}{2} - \ln\tfrac{3}{2} + \ln\tfrac{5}{4}\right) - \tfrac{1}{2}\left(\ln\left(\tfrac{3}{2}\right) - \ln\left(\tfrac{1}{2}\right) + \ln\tfrac{5}{4}\right)$$

$$= \ln\tfrac{1}{2} - \ln\tfrac{3}{2} = -\ln 3$$

THE END

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